Math 132 : Differential Topology

§ Morse functions

Consider a smooth function $f: M \to \mathbb{R}$.

If $x \in M$ is a regular point of f (i.e. if $df_x \neq 0$), we know that we can choose a local coordinate around x so that f is simply the first coordinate function.

What can we say at critical points?

Recall from calculus that a critical point x of $f: \mathbb{R}^k \to \mathbb{R}$ is called nondegenerate if the Hessian matrix $H = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)$ is nonsingular at x.

e.g. For $k=2$,

$\lambda_1, \lambda_2 > 0$ (index 0) $\quad$ $\lambda_1, \lambda_2 < 0$ (index 2) $\quad$ $\lambda_1 > 0 > \lambda_2$ (index 1)

Prop Nondegenerate critical points are isolated from the other critical points.

proof) Consider the map $\nabla f : \mathbb{R}^k \to \mathbb{R}^k$

$$x \mapsto \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_k}(x)\right).$$

Then $\nabla f(x) = 0$, since x is a critical point.

Nondegeneracy means ∇f is a local diffeomorphism at x.

Thus f has no other critical point in a small neighborhood of x. ∎

Nondegeneracy of critical points makes sense on manifolds, thanks to the following

<u>Lemma</u> Suppose $f: \mathbb{R}^k \to \mathbb{R}$ has a nondegenerate critical point at 0 and ψ is a diffeomorphism with $\psi(0)=0$.
Then $f \circ \psi$ also has a nondegenerate critical point at 0.

proof) Let $f' = f \circ \psi$. Then

$$\frac{\partial f'}{\partial x_i}(x) = \sum_\alpha \frac{\partial f}{\partial x_\alpha}(\psi(x)) \frac{\partial \psi_\alpha}{\partial x_i}(x), \quad \text{and}$$

$$\frac{\partial^2 f'}{\partial x_i \partial x_j}(0) = \sum_\alpha \sum_\beta \frac{\partial^2 f}{\partial x_\alpha \partial x_\beta}(0) \frac{\partial \psi_\alpha}{\partial x_i}(0) \frac{\partial \psi_\beta}{\partial x_j}(0) + \sum_\alpha \cancelto{0}{\frac{\partial f}{\partial x_\alpha}(0)} \frac{\partial^2 \psi_\alpha}{\partial x_i \partial x_j}(0).$$

That is, the new Hessian is $H' = (d\psi_0)^t H (d\psi_0)$.
Since H and $d\psi_0$ are nonsingular, so is H'. ∎

<u>Def</u> A function $f: M \to \mathbb{R}$ whose critical points are all nondegenerate is called a <u>Morse function</u>.

Morse functions are extremely important! They tell us great deal about the topology of M, and much of modern topology is based on this idea.
It'll take another whole course to talk about Morse theory, but let me just mention

<u>Morse lemma</u>: If $a \in M$ is a nondegenerate critical point of f, then there is a local coordinate system $(x_1, \cdots, x_m)$ around a such that
$f = f(a) + \sum h_{ij} x_i x_j$ near a, i.e. f is locally a quadratic polynomial.

As an application of Sard's theorem, let's show that the vast majority of functions are Morse functions.

<u>Thm</u> Let $f: M \to \mathbb{R}$ be a function, and suppose our manifold M sits inside $\mathbb{R}^N$. For each $a = (a_1, \cdots, a_N) \in \mathbb{R}^N$, define

$f_a := f + a_1 x_1 + \cdots + a_N x_N$, where $x_1, \cdots, x_N$ are the usual coordinate functions on $\mathbb{R}^N$.

Then, for almost every $a \in \mathbb{R}^N$, f_a is a Morse function.

proof) Let's first prove the result in $\mathbb{R}^k$:

<u>Lemma</u> Let $f: U \subset \mathbb{R}^k \to \mathbb{R}$ be a smooth function.

Then, for almost all $a = (a_1, \cdots, a_k) \in \mathbb{R}^k$,

$f_a := f + a_1 x_1 + \cdots + a_k x_k$ is a Morse function on U.

proof of Lemma)

Note that $(df_a)_p = \left(\frac{\partial f_a}{\partial x_1}(p), \cdots, \frac{\partial f_a}{\partial x_k}(p) \right) = \nabla f(p) + a$,

so p is a critical point of f_a iff $\nabla f(p) = -a$.

Moreover, it's nondegenerate iff $d(\nabla f)_p$ is nonsingular, i.e. if p is a regular point of ∇f.

Thanks to Sard, $-a$ is a regular value of ∇f for almost all $a \in \mathbb{R}^k$. ■

Now let's turn this into a global result.

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Suppose $p \in M$ and $x_1, \cdots, x_N$ are the standard coordinate functions on $\mathbb{R}^N$.
Then, some m of these coordinate functions, $x_{i_1}, \cdots, x_{i_m}$, constitute a coordinate system in a neighborhood of p.
Hence, we can cover M with countably many such open subsets U_α, and it suffices to show that f_a is Morse on U_α for almost every $a \in \mathbb{R}^N$.
For convenience, assume that $(x_1, \cdots, x_m)$ is a coordinate system on U_α.
Let $S_\alpha := \{ a \in \mathbb{R}^N \mid f_a \text{ is not Morse on } U_\alpha \}$.
Then, by the lemma, $S_\alpha \cap (\mathbb{R}^m \times \{c\})$ has measure zero in $\mathbb{R}^m$, for any $c = (c_{m+1}, \cdots, c_N)$. By Fubini, it follows that S_α has measure zero. $\blacksquare$